ALL SAINTS' COLLEGE

Mathematics Specialist
Test 2 2016

Functions

NAME: SOLUTIONS

TEACHER: MLA

50 marks

50 minutes

SCSA formulae sheets, ClassPads and a double-sided A4 sheet of notes may be used

Question 1 **[2 marks]**

Use an algebraic method to solve $|2x-4| = 10$.

For $x \geq 2$: $2x - 4 = 10$

$2x = 14$

$x = 7$ ✓

For $x < 2$:

$-(2x-4) = 10$

$2x = -6$

$x = -3$ ✓

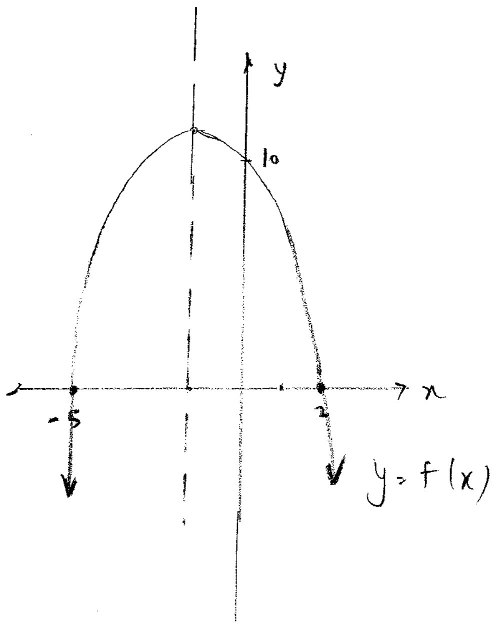
Question 2 **[4 & 1 = 5 marks]**

note $f(x) = -(x-2)(x+5)$

(a) If $f(x) = -(x^2+3x-10)$, express $|f(x)|$ and $f(|x|)$ as piecewise functions.

$$|f(x)| = \begin{cases} -(x^2+3x-10), & -5 \leq x \leq 2 \\ x^2+3x-10, & x < -5 \cup x > 2 \end{cases}$$ ✓ ✓

$$f(|x|) = \begin{cases} -(x^2+3x-10), & x \geq 0 \\ -(x^2-3x-10), & x < 0 \end{cases}$$ ✓ ✓



(b) Using your ClassPad, or otherwise, solve $|f(x)| = f(|x|)$.

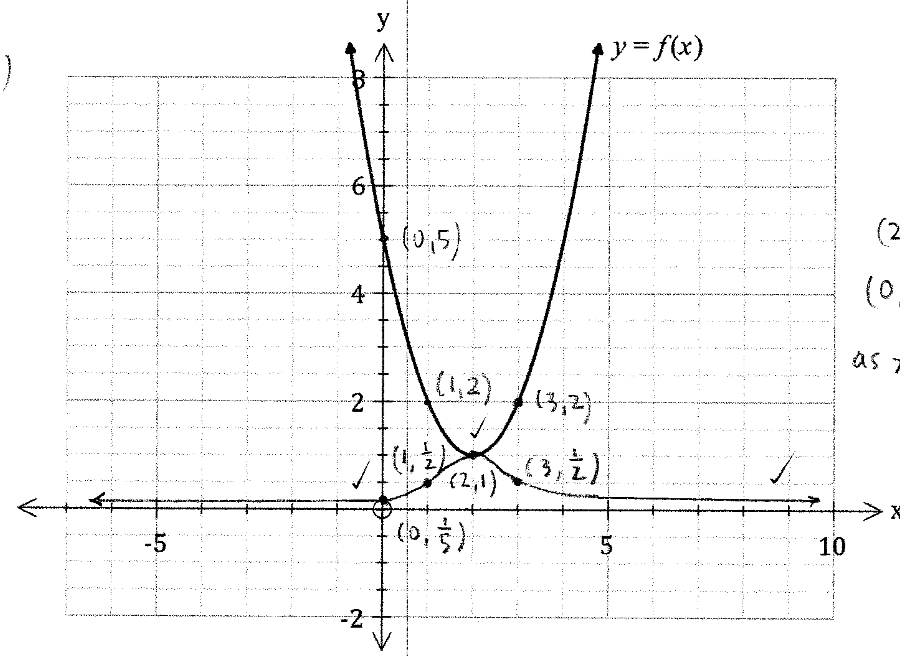
ClassPad : solve$(|f(x)| = f(|x|))$ = $\{0 \leq x \leq 2\}$ ✓

Question 3 **[3 & 3 = 6 marks]**

On the axes provided, sketch the following functions: $f(x) = (x-2)^2 + 1$

(a) $y = \cancel{f^{-1}(x)}$

$y = \frac{1}{f(x)}$



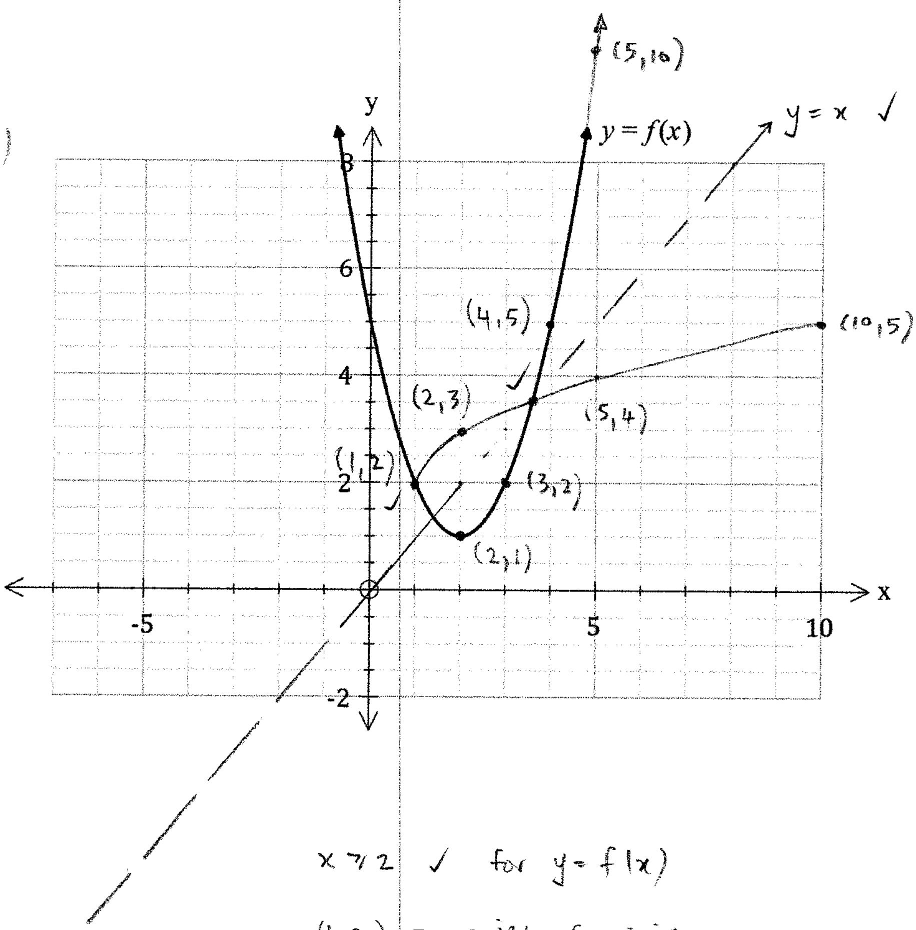
(2,1) ✓

$(0, \frac{1}{5})$ ✓

as $x \to \pm\infty$, $\frac{1}{f(x)} \to 0^+$ ✓

(b) $y = \cancel{\frac{1}{f(x)}}$

$y = f^{-1}(x)$



$x \geq 2$ ✓ for $y = f(x)$

(1,2) = point of origin

common point on $y = x$ line ✓

Question 4 **[3 marks]**

If $f(x) = 2x^2$ and $g(x) = \sqrt{2-x}$, state the rule for $fog(x)$ and find its domain and range.

$fg(x) = f(\sqrt{2-x})$

$= 2(\sqrt{2-x})^2 = 2(2-x) = 4-2x$

$x \le 2 \longrightarrow g(x) \longrightarrow y \geq 0$

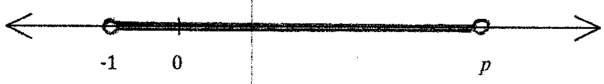
$x \geq 0 \longrightarrow f(x) \longrightarrow y \geq 0$

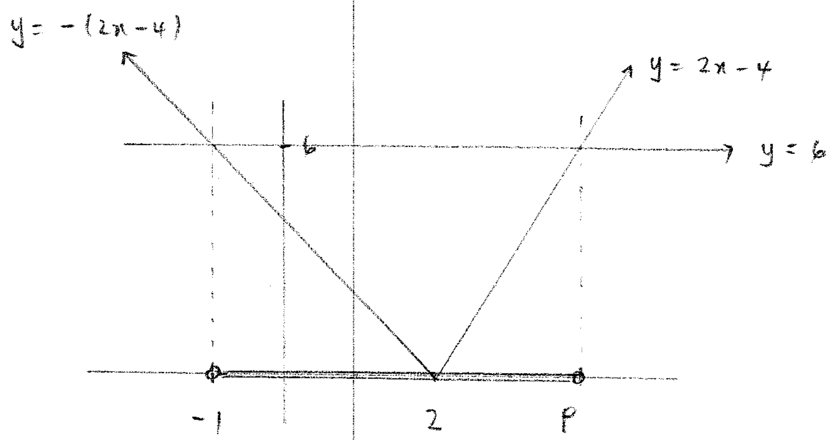
So, for $fog(x)$, domain: $\{x \in \mathbb{R} : x \le 2\}$ ✓

range: $\{y \in \mathbb{R} : y \geq 0\}$ ✓

Question 5 **[3 marks]**

With reference to the number line drawn below, determine the appropriate inequality symbol for ■, and find the values of p and k if $|2x-4|$ ■ k.





✓ ✓

Solve $2p - 4 = 6$ So, $|2x-4| < 6$

$2p = 10$

$p = 5$ ✓

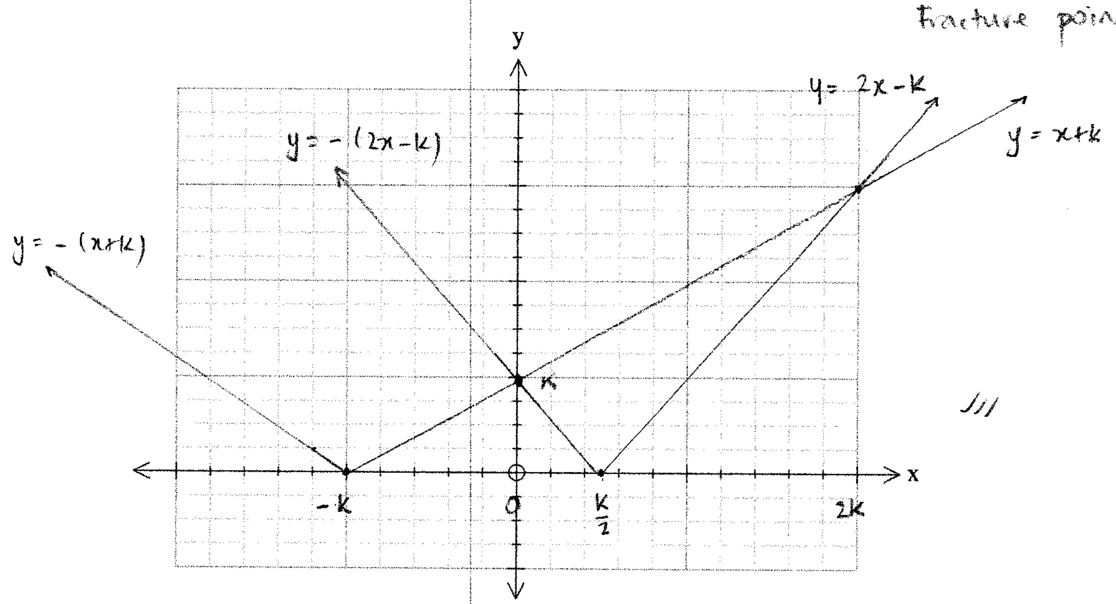
Question 6 **[3 & 3 = 6 marks]**

(a) Sketch the graphs of $f(x) = |x + k|, k > 0$ and $h(x) = |2x - k|, k > 0$

Be sure to label each graph and to identify all intercepts

Fracture points : $x = -k$

$x = \frac{k}{2}$



✓✓✓

(b) Hence, determine the value(s) of x for which $f(x) \leq h(x)$

Points of intersection :

Solve $2x - k = x + k$ and $-(2x - k) = x + k$

$x = 2k$ ✓ $\qquad -2x + k = x + k$

$x = 0$ ✓

So, $f(x) \leq h(x)$ when $\{x \in \mathbb{R} : x \leq 0 \cup x \geq 2k\}$ ✓

OR solve the following inequalities :

$x + k \leq 2x - k$ and $x + k \leq -(2x - k)$

$2k \leq x \qquad 3x \leq 0$

$x \leq 0$

Question 7 [3 marks]

Consider $f(x) = \frac{cx+d}{x+e}$, where c, d and e are integers

$f(x)$ has the following characteristics:

- vertical asymptote with equation $x = -4$
- root (zero) at $x = 4$
- intercept at $(0, 2)$

Find the values of c, d and e.

$e = 4$ ✓ $\because$ vertical asymptote $x = -4$

intercept at $(0,2) \Rightarrow 2 = \frac{d}{e}$ $\therefore$ $d = 8$ ✓

root at $(4,0) \Rightarrow 0 = 4c + 8$

$\therefore c = -2$ ✓

that is, $f(x) = \frac{8 - 2x}{x + 4}$

Question 8 **[3 & 5 = 8 marks]**

(a) Express $f(x) = x^2 + 2|x-1|$ in piecewise form.

$$f(x) = \begin{cases} x^2 + 2(x-1), & x \geq 1 \quad ✓ \\ x^2 - 2(x-1), & x < 1 \quad ✓✓ \end{cases}$$

(b) (i) Express $f(x) = |x-8| + |2-x|$ as a piecewise function.

Fracture points $x = 2, 8$

$$f(x) = \begin{cases} -(x-8) + (2-x), & x \leq 2 \\ -(x-8) - (2-x), & 2 < x \leq 8 \\ (x-8) - (2-x), & x > 8 \end{cases}$$

$$= \begin{cases} 10-2x, & x \leq 2 \quad ✓ \\ 6, & 2 < x \leq 8 \quad ✓ \\ 2x-10, & x > 8 \quad ✓ \end{cases}$$

(ii) $|x-8| + |2-x| = 4x + 4$ when $x = 1$

State the equation used to obtain this solution.

$10 - 2x = 4x + 4$ ✓✓

Question 9 **[1, 2, 1, 1, 1 & 1 = 7 marks]**

Consider $f(x) = 2 + (x-1)^2$, where $x \in \mathbb{R}$

(a) Find $f(0)$ and $f(2)$

$f(0) = 3$

$f(2) = 3$

(b) Use your answers in (a) to show that $f(x)$ does not have an inverse function

$f(0) = f(2) = 3$, but $0 \neq 2$ $\therefore$ $f(x)$ is a many-to-one function.

note. By definition, one-to-one functions exist if $f(a) = f(b)$ and $a = b$.

(c) Determine the largest possible domain for $f(x)$, consisting only of positive numbers, so that $f(x)$ has an inverse function

$\{x \in \mathbb{R} : x \geq 1\}$

(d) State the range for $f(x)$ that corresponds with your domain in (c)

$\{y \in \mathbb{R} : y \geq 2\}$

(e) Using your ClassPad, or otherwise, determine the rule for the inverse of $f(x)$ that corresponds with your domain in (c)

ClassPad: solve$(x = 2 + (y-1)^2, y)$ $\Rightarrow$ $f^{-1}(x) = 1 + \sqrt{x-2}$

(f) State the domain and range for $f^{-1}(x)$

domain : $\{x \in \mathbb{R} : x \geq 2\}$

range : $\{y \in \mathbb{R} : y \geq 1\}$

Question 10 **[5, 2 = 7 marks]**

Consider $f(x) = \frac{x^2+2x+1}{x-2}$

(a) Using your ClassPad, or otherwise, determine the following:

(i) Stationary points

$(-1, 0)$; $(5, 12)$ ✓✓

(ii) Intercept(s)

$(-1, 0)$; $(0, -\frac{1}{2})$ ✓✓

(iii) Asymptotes

$x = 2$ ✓

$y = x + 4$ (see below)

(b) Investigate the behaviour of $f(x)$ as $x \to \pm\infty$

$$\begin{array}{r|l} & x + 4 \\ \hline x - 2 & x^2 + 2x + 1 \\ & x^2 - 2x \\ \hline & 4x + 1 \\ & 4x - 8 \\ \hline & 9 \end{array}$$

note ClassPad :

$\text{propFrac}\left(\frac{x^2+2x+1}{x-2}\right)$

$= x + 4 + \frac{9}{x-2}$

$f(x) = x + 4 + \frac{9}{x-2}$

So, as $x \to \pm\infty$, $f(x) \to x + 4$ ✓✓

End of Test 2